

GRADED POLYNOMIAL IDENTITIES AND CENTRAL POLYNOMIALS OF MATRICES OVER AN INFINITE INTEGRAL DOMAIN

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ABSTRACT. Let K be an infinite integral domain and $M_n(K)$ be the algebra of all $n \times n$ matrices over K . This paper aims for the following goals:

- Find a basis for the graded identities for elementary grading in $M_n(K)$ when the neutral component and diagonal coincide;
- Describe the \mathbb{Z}_p -graded central polynomials of $M_p(K)$ when p is a prime number;
- Describe the \mathbb{Z} -graded central polynomials of $M_n(K)$.

1. INTRODUCTION

Polynomial Identity theory (PI) is an important branch of the Ring Theory. The first crucial developments in PI-theory were Kaplansky's Theorem [14] about primitive PI-algebras and the Amitsur-Levitsky Theorem [1], published in 1948 and 1950, respectively. The latter theorem is important for describing the polynomial identities of matrices.

Let K be a field and $M_n(K)$ be the algebra of all $n \times n$ matrices over K . PI-theory is used to obtain a basis for polynomial identities of $M_n(K)$. Razmyslov [21] discovered a nine-polynomial basis for the identities of $M_2(K)$ when K is a field of characteristic zero. Some years later, Drensky [11] found a minimal polynomial basis: comprising the Hall identity and the standard polynomial of degree 4. Koshlukov [18] found a basis (consisting of four identities) for the identities of $M_2(K)$ when K is an infinite field of $\text{char} K = p > 2$.

Despite these advances, the identities of $M_2(K)$ when K is an infinite field of characteristic 2 or an infinite integral domain remain unresolved.

Let K be a field of characteristic zero. In 1950 Specht [24] conjectured that every system of identities in associative algebra has a finite basis. Specht's conjecture was unsolved until the late 1980s when Kemer demonstrated its truth using the theory of \mathbb{Z}_2 -graded algebras.

Another important problem is describing the graded polynomial identities of $M_n(K)$. The \mathbb{Z}_2 -graded polynomial identities of $M_2(K)$ were described by Di Vincenzo [10], while Vasilovsky [25] described the \mathbb{Z}_n -graded polynomial identities of $M_n(K)$. A year earlier, the same author had described the \mathbb{Z} -graded identities of $M_n(K)$ [26]. Vasilovsky's results were extended by Bahturin and Drensky [4], who found the basis of the graded identities for the elementary gradings on $M_n(K)$ when the neutral component and diagonal of $M_n(K)$ coincide. Azevedo [2],[3] and Silva [23] extended the Vasilovsky's and Bahturin-Drensky's results, respectively to an infinite field.

Kaplansky [15] posed a list of open problems in Ring Theory. Among these was the question does a non-trivial central polynomial exist for $M_n(K)$ when $n \geq 3$? This question was answered by Formanek [13], and independently by Razmyslov [22].

Describing the central polynomials of $M_n(K)$ is a crucial task in PI-theory. When K is a field of characteristic zero, a set of generators may be found for the central polynomials of $M_2(K)$ [19]. Koshlukov and Colombo [9] described the central polynomials of $M_2(K)$, when K is an infinite field of characteristic $p > 2$.

The first attempts at describing graded polynomials of $M_n(K)$ were made by Brandão Júnior [8]. Assuming an infinite ground field K , he described the \mathbb{Z}_n -graded central polynomials of $M_n(K)$ when $\text{char}K \nmid n$, as well as, the \mathbb{Z}_p -graded central polynomials of $M_p(K)$ (where $\text{char}K = p > 2$) and the \mathbb{Z} -graded central polynomials of $M_n(K)$.

Few reports of graded identities of $M_n(K)$ exist in the literature, when K is an infinite integral domain. Brandão Júnior, Koshlukov and Krasilnikov [7] detailed a basis for the \mathbb{Z}_2 -graded identities of $M_2(K)$. They also described a basis for the \mathbb{Z}_2 -graded central polynomials of $M_2(K)$.

In this paper, we combine the methods of [2], [3], [4], [8], [12], [23], [25] and [26]. This paper aims for the following goals:

- Find a basis for the graded identities for elementary grading in $M_n(K)$ when the neutral component and diagonal coincide;
- Describe the \mathbb{Z}_p -graded central polynomials of $M_p(K)$ when p is a prime number;
- Describe the \mathbb{Z} -graded central polynomials of $M_n(K)$.

In these three situations, K is an infinite integral domain.

2. PRELIMINARIES

Let K be a fixed unital associative and commutative ring. We assume that all modules are left-modules over K and all (unital associative) algebras are considered over K . We assume that all ideals are bilateral ideals. The set $\{1, \dots, n\}$ is denoted by \hat{n} . Moreover, G denotes an arbitrary group and $\mathbb{N} = \{0, 1, 2, \dots, n, \dots\}$ is the set of natural numbers. Here, S_n denotes the group of permutations on \hat{n} . H_n denotes the subgroup of S_n generated by $(12 \dots n)$.

Let X be a countable set of variables and $K\langle X \rangle$ be the free associative ring freely generated by X . Let A be an algebra over K and let $Z(A)$ be the center of A . A polynomial $f(x_1, \dots, x_n) \in K\langle X \rangle$ is called an ordinary polynomial identity (respectively an ordinary central polynomial) for A if $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in A$ (respectively $f(0, \dots, 0) = 0$ and $f(a_1, \dots, a_n) \in Z(A)$). The algebra A is a PI-algebra if there exists $f \in K\langle X \rangle$ satisfying the following conditions:

- : f is an ordinary polynomial identity for A ;
- : Some coefficient in the highest-degree homogeneous component of f equals to 1.

The set of ordinary polynomial identities (respectively ordinary central polynomials) of A is an ideal (respectively submodule) of $K\langle X \rangle$ that is invariant under all endomorphisms of $K\langle X \rangle$. The ideals (respectively submodules) of $K\langle X \rangle$ that

are invariant under all endomorphisms of $K\langle X \rangle$ are called T -ideals (respectively T -spaces).

Clearly, the intersection of a family of T -ideals (respectively T -spaces) of $K\langle X \rangle$ is also a T -ideal (respectively a T -space). Let S be a non-empty set of $K\langle X \rangle$. The T -ideal (respectively T -space) generated by S , denoted $\langle S \rangle_T$ (respectively $\langle S \rangle^T$), is the intersection of all T -ideals (respectively T -spaces) containing S .

An algebra A over K is G -graded when there exist K -submodules $\{A_g\}_{g \in G} \subset A$ such that:

$$A = \bigoplus_{g \in G} A_g \quad (1);$$

$$A_g A_h \subset A_{gh} \text{ for all } g, h \in G \quad (2).$$

Each submodule A_g is called the homogeneous component of G -degree g and its non-zero elements are homogeneous elements of G -degree g . Moreover, if a is a homogeneous element, its G -degree is denoted by $\alpha(a)$. We denote the identity element of G by e . The support of A , with respect to the grading $\{A_g\}_{g \in G}$, is the following subset of G :

$$\text{Supp}_G(A) := \{g \in G \mid A_g \neq \{0\}\}.$$

Let $\{X_g \mid g \in G\}$ be a family of disjoint countable sets indexed by G and let $X = \bigcup_{g \in G} X_g$. $K\langle X \rangle_g$ is the K -submodule of $K\langle X \rangle$ spanned by $m = x_{j_1} \cdots x_{j_k}$ such that $\alpha(m) = g$. The decomposition $K\langle X \rangle = \bigoplus_{g \in G} K\langle X \rangle_g$ is a G -grading, whereby $K\langle X \rangle$ is the G -graded free associative ring freely generated by X . A monomial is a variable or a product of variables in X .

Let $m = x_{i_1} \cdots x_{i_l}$ be a monomial of $K\langle X \rangle$. We denote by $h(m)$ the l -tuple $(\alpha(x_{i_1}), \dots, \alpha(x_{i_l}))$.

An endomorphism ϕ of $K\langle X \rangle$ is called G -graded endomorphism when $\phi(K\langle X \rangle_g) \subset K\langle X \rangle_g \forall g \in G$. When a graded ideal (respectively a graded submodule) $I \subset K\langle X \rangle$ is invariant under all G -graded endomorphisms of $K\langle X \rangle$ is called a T_G -ideal (respectively a T_G -space). A graded polynomial $f(x_1, \dots, x_n) \in K\langle X \rangle$ is a G -graded polynomial identity for A (respectively a G -graded central polynomial for A) if $f(a_1, \dots, a_n) = 0$ for all $a_i \in A_{\alpha(x_i)}$, $i = 1, \dots, n$ ($f(0, \dots, 0) = 0$ and $f(a_1, \dots, a_n) \in Z(A)$ for all $a_i \in A_{\alpha(x_i)}$, $i = 1, \dots, n$).

The set of all G -graded identities of A (respectively all G -graded central polynomials of A) is denoted by $T_G(A)$ (respectively $C_G(A)$). Clearly, $T_G(A)$ (respectively $C_G(A)$) is a T_G -ideal (respectively a T_G -space and a subalgebra) and the intersection of a family of T_G -ideals (respectively T_G -spaces) of $K\langle X \rangle$ is also a T_G -ideal (respectively a T_G -space). The T_G -ideal (respectively T_G -space) generated by a non-empty set S of $K\langle X \rangle$ is defined as in the ordinary case. The T_G -ideal (respectively a T_G -space) generated by S is denoted by $\langle S \rangle_{T_G}$ (respectively $\langle S \rangle^{T_G}$). A graded polynomial f is said to be a consequence of $S \subset K\langle X \rangle$ if $f \in \langle S \rangle_{T_G}$ (or equivalently, that f follows from S). A set $S \subset K\langle X \rangle$ is called a basis for the graded identities (respectively the graded central polynomials) of A if $T_G(A) = \langle S \rangle_{T_G}$ (respectively $C_G(A) = \langle S \rangle^{T_G}$).

The matrix unit $e_{ij} \in M_n(K)$ contains 1 as the only a non-zero value in the i -th row and j -th column. Given an n -tuple $\vec{g} = (g_1, \dots, g_n) \in G^n$, a G -grading is determined in $M_n(K)$ by stipulating that e_{ij} is homogeneous of G -degree $g_i^{-1}g_j$. These gradings are elementary and we say that $M_n(K)$ possesses an elementary grading induced by \vec{g} . We equipped $M_n(K)$ with an elementary grading induced by an n -tuple of distinct elements from G . The set $\{g_1, \dots, g_n\}$ is denoted by G_n .

Below we provided two important examples of elementary gradings on $M_n(K)$ whose neutral component and diagonal coincide:

- : \mathbb{Z}_n -canonical grading (or \mathbb{Z}_n -grading): when $G = \mathbb{Z}_n$ and the n -tuple \bar{g} is $(\bar{1}, \bar{2}, \dots, \overline{n-1}, \bar{n})$;
- : \mathbb{Z} -canonical grading (or \mathbb{Z} -grading): when $G = \mathbb{Z}$ and the n -tuple \bar{g} is $(1, 2, \dots, n-1, n)$.

3. SILVA'S GENERIC MODEL

Generic models have an important role in PI (see for instance [5], [6] and [20]). In this section, we recall Silva's Generic Model, as described in [23]. Let G be an arbitrary group and let $\bar{g} = (g_1, \dots, g_n) \in G^n$ be an n -tuple of distinct elements from G . We consider the algebra $M_n(K)$ to be equipped with the elementary grading induced by \bar{g} . For each $h \in G$, let $Y_h = \{y_{h,i}^k | 1 \leq k \leq n; i \geq 1\}$ be a countable set of commuting variables and let $Y = \bigcup_{h \in G} Y_h$. Let $\Omega = K[Y]$ denote the polynomial ring with commuting variables in Y . Let $M_n(\Omega)$ be the algebra of all $n \times n$ matrices over Ω . This algebra may be equipped with the elementary grading induced by \bar{g} as $M_n(K)$. Let G_n denote the set $\{g_1, \dots, g_n\}$.

Definition 3.1. Let h be an element of G . Let L_h denote, the set of all indices $k \in \hat{n}$ such that $g_k h \in G_n$. Let s_h^k denote the index determined by $g_{s_h^k} := g_k h$.

Let $\mathbf{h} = (h_1, \dots, h_m) \in G^m$. L_h defines (the set of indices associated with the m -tuple (h_1, \dots, h_m)) the subset of \hat{m} such that its elements satisfy the following property:

$$g_k h_1 \cdots h_i \in G_n \quad \forall i \in \hat{m}.$$

We define a sequence $(s_1^k, \dots, s_{m+1}^k)$ (the sequence associated with \mathbf{h} is determined by k) inductively by setting:

- : 1) $s_1^k = k$;
- : 2) $s_l^k : g_{s_l^k} = g_k h_1 \cdots h_{l-1} \quad \forall l \in \{2, \dots, m+1\}$

A generic matrix of G -degree h is a homogeneous element of $M_n(\Omega)$ the following type:

$$A_i^h = \sum_{k \in L_h} y_{h,i}^k e_{k, s_h^k}.$$

The G -graded subalgebra of $M_n(\Omega)$ generated by the generic matrices is called the algebra of the generic matrices which we denote by R . Notice that $\text{Supp}_G(R) = \text{Supp}_G(M_n(K))$.

The next lemma is an important computational result. Its proof is the immediate consequence of a multiplication table of matrix units.

Lemma 3.2. ([23], Lemma 3.5) If L is the set of indices associated with the q -tuple (h_1, \dots, h_q) in G^q and $s_k = (s_1^k, \dots, s_{q+1}^k)$ denotes the corresponding sequence determined by $k \in L$, then

$$A_{i_1}^{h_1} \cdots A_{i_q}^{h_q} = \sum_{k \in L} w_k e_{s_1^k, s_{q+1}^k}$$

which $w_k = y_{h_1, i_1}^{s_1^k} y_{h_2, i_2}^{s_2^k} \cdots y_{h_q, i_q}^{s_q^k}$.

Definition 3.3. Let $f(x_1, \dots, x_n)$ be a polynomial of $K\langle X \rangle$ and let $A_1 \in R_{\alpha(x_1)}, \dots, A_n \in R_{\alpha(x_n)}$. $f(A_1, \dots, A_n)$ denotes the result of replacing for the corresponding elements of R .

The next lemma is the same in ([23], Lemma 4.5).

Lemma 3.4. *Let $M(x_1, \dots, x_q)$ and $N(x_1, \dots, x_q)$ be two monomials of $K\langle X \rangle$ that start with the same variable. Let $m(x_1, \dots, x_q)$, $n(x_1, \dots, x_q)$ be two monomials obtained from M and N respectively by deleting the first variable. If there exist matrices A_1, \dots, A_q , such that $M(A_1, \dots, A_q)$ and $N(A_1, \dots, A_q)$ have, in the same position, the same non-zero entry, then the matrices $m(A_1, \dots, A_q)$ and $n(A_1, \dots, A_q)$ also have, in the same position, the same non-zero entry.*

Proof. It is the immediate consequence of Lemma 3.2. \square

Remark 3.5. ([23], Corollary 3.7) *Notice that if m_1 and m_2 are two monomials such that $h(m_1) = h(m_2)$, then $m_1 \in T_G(R)$ if and only if $m_2 \in T_G(R)$.*

Remark 3.6. *Let m be a monomial. Notice that $m \in T_G(R)$ if and only if $m \in T_G(M_n(K))$.*

The next three lemmas can be proved by elementary algebraic methods.

Lemma 3.7. *Let $f \in T_G(R)$. Then all multi-homogeneous components of f are elements of $T_G(R)$.*

Lemma 3.8. *Let $f \in C_G(R)$. Then all multi-homogeneous components of f are elements of $C_G(R)$.*

Lemma 3.9. *Let $m(x_1, \dots, x_q) = x_{i_1} \cdots x_{i_r}$ and $n(x_1, \dots, x_q)$ be two monomials such that the matrices $n(A_1, \dots, A_q)$ and $m(A_1, \dots, A_q)$ have, at some position, the same non-zero entry. Then $m - n$ is a multi-homogeneous polynomial.*

Observe that $T_G(R) \subset T_G(M_n(K))$. The proof of the next lemma is similar to that proof in ([2], Lemma 3).

Lemma 3.10. *Let K be an infinite integral domain. Then $T_G(M_n(K)) = T_G(R)$.*

Corollary 3.11. *Let K be an infinite integral domain. Then $C_G(M_n(K)) = C_G(R)$.*

4. SOME GRADED IDENTITIES OF R AND TYPE 1-MONOMIALS

In this section, we present some graded identities for elementary grading in R when the neutral component and diagonal coincide. Notice that R is equipped with the elementary grading induced by an n -tuple of pairwise elements from G if and only if R_e and diagonal coincide.

The next lemma was proved by Bahturin and Drensky in ([4], Lemma 4.1) for full algebra of n by n matrices over a field of characteristic zero.

Lemma 4.1. *The following graded polynomials are G -graded polynomial identities for R :*

- $x_1x_2 - x_2x_1$ when $\alpha(x_1) = \alpha(x_2) = e$ (1);
- $x_1x_2x_3 - x_3x_2x_1$ when $\alpha(x_1) = \alpha(x_3) = (\alpha(x_2))^{-1} \neq e$ (2);
- x_1 when $R_{\alpha(x_1)} = \{0\}$ (3).

Proof. It follows from Lemma 3.2 and the proof of Lemma 4.1 in [4]. \square

Definition 4.2. *Let J be the T_G -ideal generated by (1), (2) and (3). Let J_1 be the T_G -ideal generated by (1) and (2) only.*

Definition 4.3. Let $m = x_{i_1} \cdots x_{i_q}$ be a G -graded monomial. An element $n \in K\langle X \rangle$ is called a subword of m when there exist $j \in \{0, \dots, q\}$ and $l \in \mathbb{N}$, where $j + l \leq q$, such that:

$$n = x_{i_j} \cdots x_{i_{j+l}}.$$

Likewise, the monomial n is termed a proper subword of m when n is a subword of m , $n \neq m$ and $n \neq 1$.

Definition 4.4. Let $m = x_{i_1} \cdots x_{i_l}$ be a G -graded monomial. This monomial is called a Type 1-monomial when the G -degree of all its non-empty subwords are elements of $\text{Supp}_G(R)$.

([4]) shows an example of G -graded Type 1-monomial identities of $M_n(K)$ (see Example 4.7, [4]).

In this paper, the following lemma is useful.

Lemma 4.5. Let m be a multilinear Type 1-monomial. Let \overline{m} be the polynomial obtained from m by deleting the variables of G -degree e by one. Then $m \in T_G(R)$ if and only if $\overline{m} \in T_G(R)$.

The proof of the following lemma is left as an exercise.

Lemma 4.6. Let $m \in T_G(R)$ be a monomial. If m is not a Type 1-monomial identity, then m follows from (3).

5. TYPE 1-MONOMIAL IDENTITIES OF R

This section describes the monomial Type 1-identities for elementary grading in R when the neutral component and diagonal coincide. The number s denotes $|\text{Supp}_G(R)|$ and λ denotes the number $[s + 1][(s + 1)(\sum_{i=1}^s (s - 1)^i) + 1]$.

Definition 5.1. Let $m = x_{i_1} \cdots x_{i_q}$. Let k, l be two positive integers such that $1 \leq k \leq l \leq q$. We define the monomial $m^{[k, l]}$ obtained from m by deleting the $k - 1$ first variables and the $q - l$ last variables.

Definition 5.2. Let \mathcal{S} denote the set of all sequences with elements in $\text{Supp}_G(R)$ whose length is less than $s + 1$. Let $\mathcal{A} = \{(g_1, \dots, g_m) \in \mathcal{S} | g_1 \cdots g_m = e\}$.

Remark 5.3. Notice that $|\mathcal{S}| = \sum_{i=1}^s s^i$.

Definition 5.4. A monomial $m = x_{i_1} \cdots x_{i_l}$ is called a Type 2-monomial when there exist $a \in \mathbb{N} - \{0\}$, $p_1, p_2 \in \widehat{l}$ such that $1 \leq p_1 < p_1 + a < p_2 < p_2 + a \leq l$ and:

$$\begin{aligned} &: \alpha(x_{i_{p_1}} \cdots x_{i_{p_1+a}}) = \alpha(x_{i_{p_1+a+1}} \cdots x_{i_{p_2-1}}) = \alpha(x_{i_{p_2}} \cdots x_{i_{p_2+a}}) = e; \\ &: h(x_{i_{p_1}} \cdots x_{i_{p_1+a}}) = h(x_{i_{p_2}} \cdots x_{i_{p_2+a}}). \end{aligned}$$

Definition 5.5. Let $m = x_{i_1} \cdots x_{i_l}$ be a monomial. It is called a Type 3-monomial when it does not have a proper subword of G -degree e . Otherwise, it is called a Type 4-monomial.

Corollary 5.6. Let $m = x_{i_1} \cdots x_{i_l}$ be a Type 1-monomial without variables of G -degree e . If $l > s$, then m is a Type 4-monomial.

Proof. Let $\beta(t) = \alpha(m^{[1, t]})$ be a function with domain \widehat{l} and codomain $\text{Supp}_G(R)$. For the hypothesis, $l > s$. Consequently, according to the Pigeonhole Principle, there exists $1 \leq t_1 < t_2 \leq l$ such that $\beta(t_1) = \beta(t_2)$. Note that $t_1 + 1 < t_2$ because m does not have variable of G -degree e . So, $m^{[t_1+1, t_2]}$ satisfies the thesis statement of the corollary. \square

Lemma 5.7. *Let S be a multiset formed by elements of $(\sum_{i=1}^s (s-1)^i)$. If $|S| \geq (s+1)(\sum_{i=1}^s (s-1)^i) + 1$, then there exists $i \in (\sum_{i=1}^s (s-1)^i)$ such that this positive integer repeats, at least, $s+2$ times in S .*

Proof. It is the immediate consequence of the Pigeonhole Principle. \square

Lemma 5.8. *Let $m = x_1 \cdots x_r$ be a multilinear Type 1-monomial without variables of G -degree e . If the ordinary degree of m is greater than or equal to λ , then it is a Type 2-monomial.*

Proof. Let m be a multilinear Type 1-monomial of $K\langle X \rangle$ without variables of G -degree e whose ordinary degree is greater than or equal to $(s+1)^2(\sum_{i=1}^s (s-1)^i) + (s+1)$ and a denotes the number $(s+1)(\sum_{i=1}^s (s-1)^i) + 1$.

Let:

$$m_1 = m^{[1, s+1]}, m_2 = m^{[s+2, 2(s+1)]}, \dots, m_a = m^{[(a-1)(s+1)+1, a(s+1)]}.$$

By Corollary 5.6, for each m_i , there is a proper subword of G -degree e and ordinary degree less than or equal to s .

Let $\gamma : \{m_1, \dots, m_a\} \rightarrow \mathcal{A}$ be a relation that assigns: $(g_1, \dots, g_{n_1}) \in \gamma(m_i)$ if, and only if, there exists a subword of m_i of ordinary degree n_1 , $m_{i,1}$, such that $h(m_{i,1}) = (g_1, \dots, g_{n_1})$. By Lemma 5.7, there exist subwords $m_{i_1,1}, \dots, m_{i_{s+2},1} \in \{\gamma(m_1), \dots, \gamma(m_a)\}$ of $m_{i_1}, \dots, m_{i_{s+2}}$ ($i_1 < \dots < i_{s+2}$) such that $h(m_{i_1,1}) = \dots = h(m_{i_{s+2},1})$. By Pigeonhole Principle, there exist $k, k+l \in \{1, \dots, s+2\}$ such that the subword of m ($m_{i_k, i_{k+l}, 1}$), between $m_{i_k,1}$ and $m_{i_{k+l},1}$, with G -degree e .

Therefore, $m_{i_k,1} m_{i_k, i_{k+l}, 1} m_{i_{k+l},1}$ is a Type 2-monomial. So, m is Type 2-monomial as well. \square

Definition 5.9. *We denote by U the T_G -ideal generated by the following identities of R :*

- $x_1 x_2 - x_2 x_1$ when $\alpha(x_1) = \alpha(x_2) = e$ (1);
- $x_1 x_2 x_3 - x_3 x_2 x_1$ when $\alpha(x_1) = \alpha(x_3) = (\alpha(x_2))^{-1} \neq e$ (2);
- x_1 when $R_{\alpha(x_1)} = \{0\}$ (3);
- The multilinear Type 1-monomial identities whose ordinary degrees are less than or equal to λ (4).

Recall that if m is a monomial, then $m \in T_G(R)$ if and only if $m \in T_G(M_n(K))$ (Remark 3.6). In the next lemma we follow an idea of ([4], Proposition 4.2).

Lemma 5.10. *Let $m = x_1 \cdots x_q$ be ($q > \lambda$) a multilinear monomial. If m is a Type 1-monomial identity for R , then m follows from (4).*

Proof. Let $m = x_1 \cdots x_q$ be a multilinear Type 1-monomial identity for R where $q \geq \lambda + 1$. We may suppose without any loss of generality that $\alpha(x_i) \neq e$ for all $i \in \hat{q}$ (Lemma 4.5). The proof is made by induction on q . Suppose that $q = \lambda + 1$. According to Lemma 5.8, m is a Type 2-monomial.

Here, we use the same notation as in Lemma 5.8. If $x_{p_1} \cdots x_{p_2+a}$ is a graded monomial identity for R , then m is a consequence of the monomial identities of type (4). Thus, we may suppose that $x_{p_1} \cdots x_{p_2+a} \notin T_G(R)$. Let $\hat{m} = m^{[1, p_1-1]} m^{[p_1+a+1, q]}$. If \hat{m} is a monomial identity for R , then m is a consequence of \hat{m} that is a Type 1-monomial. Suppose for contradiction that \hat{m} is not a polynomial identity for R . We may suppose without loss of generality that $p_1 + a + 1 < p_2$ and $q \geq p_2 + a +$

1. Therefore, there exist the matrix units $e_{l_1 k_1} \in M_n(K)_{\alpha(x_1)}, \dots, e_{l_{p_1-1} k_{p_1-1}} \in M_n(K)_{\alpha(x_{p_1-1})}, e_{l_{p_1+a+1} k_{p_1+a+1}} \in M_n(K)_{\alpha(x_{p_1+a+1})}, \dots, e_{l_q k_q} \in M_n(K)_{\alpha(x_q)}$ such that $(e_{l_1 k_1} \cdots e_{l_{p_1-1} k_{p_1-1}}) \cdot (e_{l_{p_1+a+1} k_{p_1+a+1}} \cdots e_{l_q k_q}) \neq 0$.

Note that $k_{p_1-1} = l_{p_1+a+1} = k_{p_2-1} = l_{p_2} = k_{p_2+a}$. In this form, consider the following evaluation in m : $x_i = e_{l_i k_i} \quad \forall \quad i \in \widehat{q} - \{p_1, \dots, p_1 + a\}, x_{l_{p_1+j} k_{p_1+j}} = e_{l_{p_2+j} k_{p_2+j}} \quad \forall \quad j \in \{0, 1, \dots, a\}$.

Thus $(e_{l_1 k_1} \cdots e_{l_{p_1-1} k_{p_1-1}}) \cdot (e_{l_{p_2} k_{p_2}} \cdots e_{l_{p_2+a} k_{p_2+a}}) \cdot (e_{l_{p_1+a+1} k_{p_1+a+1}} \cdots e_{l_q k_q}) \neq 0$.

This is a contradiction, because $m \in T_G(R)$. By induction on q , the result follows. \square

Lemma 5.11. *If m is a Type 1-monomial identity of R , then m follows from (4).*

Proof. It follows from Remark 3.5 and Lemma 5.10. \square

6. THE MAIN RESULT

The next lemma follows an idea of ([2], Lemma 5), ([3], Lemma 5), ([25], Lemma 4) and ([23], Lemma 4.6).

Lemma 6.1. *Let $m(x_1, \dots, x_q)$ and $n(x_1, \dots, x_q)$ be two monomials such that the matrices $n(A_1, \dots, A_q)$ and $m(A_1, \dots, A_q)$ have in same position the same non-zero entry. Then:*

$$m(x_1, x_2, \dots, x_q) \equiv n(x_1, x_2, \dots, x_q) \pmod{J_1}.$$

Proof. Let $m = x_{i_1} \cdots x_{i_r}$. According to Lemma 3.9, $m - n$ is a multi-homogeneous polynomial. Let m_1 and n_1 be two multilinear monomials (with the same variables) such that $h(m_1) = h(m)$ and $h(n_1) = h(n)$. Note that, it is enough to prove:

$$m_1 \equiv n_1 \pmod{J_1}.$$

We may suppose that $m_1 = x_1 \cdots x_r$. Therefore, there exists $\sigma \in S_r$ such that $n_1 = x_{\sigma(1)} \cdots x_{\sigma(r)}$. By this hypothesis, there is a position $(i, j) \in \widehat{n} \times \widehat{n}$ such that $e_{i_1} m_1(A_1, \dots, A_q) e_{j_1} = e_{i_1} n_1(A_1, \dots, A_q) e_{j_1} \neq 0$.

Suppose that the entry of $m_1(A_1, \dots, A_q)$, in the position (i, j) is: $y_{\alpha(x_1), 1}^{q_1} \cdots y_{\alpha(x_r), r}^{q_r}$ where $q_2, \dots, q_r \in \widehat{n}$ and $q_1 = i$. Therefore:

$$e_{q_1} s_{\alpha(x_1)}^{q_1} \cdots e_{q_r} s_{\alpha(x_r)}^{q_r} = e_{q_{\sigma(1)}} s_{\alpha(x_{\sigma(1)})}^{q_{\sigma(1)}} \cdots e_{q_{\sigma(r)}} s_{\alpha(x_{\sigma(r)})}^{q_{\sigma(r)}} = e_{ij}.$$

In this form, there exist matrix units $e_{i_1 j_1} \in M_n(K)_{\alpha(x_1)}, \dots, e_{i_r j_r} \in M_n(K)_{\alpha(x_r)}$ having the following property:

$$e_{i_1 j_1} \cdots e_{i_r j_r} = e_{i_{\sigma(1)} j_{\sigma(1)}} \cdots e_{i_{\sigma(r)} j_{\sigma(r)}} \neq 0.$$

So, $i_1 = i_{\sigma(1)}, j_r = j_{\sigma(r)}$ and $\alpha(m_1) = \alpha(n_1) = g_i^{-1} g_j$.

In the following steps, we will be use an induction on r . If $r = 1$, the proof is obvious.

- Step 1: Suppose that $\sigma(1) = 1$. In this case, the monomials m_1 and n_1 start with the same variable. Let m_2 and n_2 be two monomials obtained from m_1 and n_1 respectively by deleting the first variable. By Lemma 3.4, $m_2(A_1, \dots, A_q)$ and $n_2(A_1, \dots, A_q)$ have, in the same position, the same non-zero entry. Hence, by induction hypothesis, $m_2 \equiv n_2$ modulo J . Consequently, $m_1 \equiv n_1$ modulo J_1 as required.

: Step 2: Suppose that $\sigma(1) > 1$. Let t be the least positive integer such that $\sigma^{-1}(t+1) < \sigma^{-1}(1) \leq \sigma^{-1}(t)$. We define: $k_1 := \sigma^{-1}(t+1)$, $k_2 := \sigma^{-1}(1)$ and $k_3 = \sigma^{-1}(t)$. Note that: $\sigma(k_1) = t+1$, $\sigma(k_2) = 1$, $\sigma(k_3) = t$, $i_{\sigma(c+1)} = j_{\sigma(c)}$ and $i_{c+1} = j_c$ for all $c \in \widehat{r-1}$. It is clear that:

$$n_1 = x_{\sigma(1)} \cdots x_{\sigma(r)} = n_1^{[1, k_1-1]} n_1^{[k_1, k_2-1]} n_1^{[k_2, k_3]} n_1^{[k_3+1, r]}.$$

Likewise:

$$\begin{aligned} \alpha(n_1^{[1, k_1-1]}) &= g_{i_{\sigma(1)}}^{-1} g_{j_{\sigma(k_1-1)}} = g_{i_1}^{-1} g_{i_{\sigma(k_1)}} = g_{i_1}^{-1} g_{i_{t+1}}; \\ \alpha(n_1^{[k_1, k_2-1]}) &= g_{i_{\sigma(k_1)}}^{-1} g_{j_{\sigma(k_2-1)}} = g_{i_{t+1}}^{-1} g_{i_{\sigma(k_2)}} = g_{i_{t+1}}^{-1} g_{i_1}; \\ \alpha(n_1^{[k_2, k_3]}) &= g_{i_{\sigma(k_2)}}^{-1} g_{j_{\sigma(k_3)}} = g_{i_1}^{-1} g_{j_t} = g_{i_1}^{-1} g_{i_{t+1}}. \end{aligned}$$

Thus, by the identities (1) and (2), it is possible to conclude that:

$$n_1 \equiv n_1^{[k_2, k_3]} n_1^{[k_1, k_2-1]} n_1^{[1, k_1-1]} n_1^{[k_3+1, r]} \pmod{J_1}.$$

Conclusion: n_1 is a congruent monomial that starts with the same variable of m_1 . Repeating the arguments of the first case, we conclude that $m_1 \equiv n_1$ modulo J_1 . □

Lemma 6.2. *Let G be a finite group of order n . Then R does not satisfy a G -graded monomial identity.*

Proof. According to Remarks 3.5 and 3.6, it is sufficient to prove that $M_n(K)$ does not satisfy a multilinear monomial identity $x_1 \dots x_l$.

It is clear that $\text{Supp}_G(M_n(K)) = G$. So, it is enough to prove that $M_n(K)$ does not satisfy a Type-1 multilinear monomial identity. If $l = 1$, the proof is obvious. The proof is made by induction on l . According to the hypothesis of induction, there exist matrix units $e_{i_2 j_2} \in (M_n(K))_{\alpha(x_2)}, \dots, e_{i_l j_l} \in (M_n(K))_{\alpha(x_l)}$ such that $e_{i_2 j_2} \cdots e_{i_l j_l} = e_{i_2 j_l}$. Notice that there exists $g_k \in \{g_1, \dots, g_n\}$ such that $g_k^{-1} g_i = \alpha(x_1)$. So, $e_{k i_2} \cdots e_{i_l j_l} \neq 0$. The proof of Lemma 6.2 is complete. □

Lemma 6.3. *If R does not satisfy a monomial identity, then $T_G(R) = J_1$.*

Proof. Suppose for contradiction there exists $f(x_1, \dots, x_t) = \sum_{i=1}^l \lambda_i m_i \in T_G(R) - J_1$, where for all $i \in \widehat{l}$: $\lambda_i \in K - \{0\}$, m_i is a monomial. According to Lemma 3.7, we may suppose that f is a multi-homogeneous polynomial. Moreover, it can be supposed that l is the least positive integer with the following set:

$$B = \{q \in \mathbb{N} \mid \sum_{i=1}^q \gamma_i n_i(x_1, \dots, x_t) \in T_G(R) - J_1; \gamma_i \in K - \{0\} \text{ for all } i \in \widehat{q}\}.$$

It is clear that $m_i(A_1, \dots, A_t) \neq 0$, for $i = 1, \dots, l$, because R does not satisfy a monomial identity. Furthermore, there exists $k \in \{2, \dots, l\}$ such that:

$$m_1(A_1, \dots, A_t) \text{ and } m_k(A_1, \dots, A_t)$$

have, in the same position the same non-zero entry. Thus by Lemma 6.1, it follows that $m_1 \equiv m_k$ modulo J_1 . Consequently, $h = f + \lambda_k(m_1 - m_k) \in T_G(R) - J_1$. The contraction, in addition the number of non-zero summands in h is less than l . □

Corollary 6.4. *Let K be an infinite integral domain. The \mathbb{Z}_n -graded polynomial identities of $M_n(K)$ follow from:*

- $x_1 x_2 - x_2 x_1$ when $\alpha(x_1) = \alpha(x_2) = \bar{0}$ (1);
- $x_1 x_2 x_3 - x_3 x_2 x_1$ when $\alpha(x_1) = \alpha(x_3) = -(\alpha(x_2)) \neq \bar{0}$ (2).

Proof. It follows from Lemmas 6.2 and 6.3. \square

Following word for word the work of Vasilovsky in [26] (see Lemma 1, Lemma 3 and Corollary 4), we have the following lemma:

Lemma 6.5. *Let K be an infinite integral domain and $m = x_1 \cdots x_l$ be a multilinear Type 1-monomial. Then $m \notin T_{\mathbb{Z}}(M_n(K))$.*

Corollary 6.6. *Let K be an infinite integral domain and m be a Type 1-monomial. Then $m \notin T_{\mathbb{Z}}(M_n(K))$.*

Following word for word the proof of Lemma 6.3, we have the following lemma:

Lemma 6.7. *If R does not satisfy a Type 1-monomial identity, then $T_G(R) = J$.*

Corollary 6.8. *Let K be an infinite integral domain. The \mathbb{Z} -graded polynomial identities of $M_n(K)$ follow from:*

- $x_1x_2 - x_2x_1$ when $\alpha(x_1) = \alpha(x_2) = 0$ (1);
- $x_1x_2x_3 - x_3x_2x_1$ when $\alpha(x_1) = \alpha(x_3) = -(\alpha(x_2)) \neq 0$ (2);
- x_1 when $|\alpha(x_1)| \geq n$ (3).

Proof. It follows from Corollary 6.6 and Lemma 6.7. \square

Now, we present the main result of this paper.

Theorem 6.9. *Let G be an arbitrary group. Then $T_G(R) = U$. If K is an infinite integral domain, then $T_G(M_n(K)) = U$.*

Proof. According to Lemmas 4.6 and 5.11, if m is a monomial identity of R , then m is consequence of (3) or (4). Thus, it is sufficient to imitate the proof of Lemma 6.3 and replace J_1 with U , early in the proof. \square

7. MATRIX-UNITS GRADED IDENTITIES OF $M_n(K)$ OVER AN INFINITE INTEGRAL DOMAIN

The algebra $M_n(K)$ has a natural grading by MU_n , the semigroup of matrix units of class n .

Definition 7.1. *Let $MU_n = \{(i, j) \in \widehat{n} \times \widehat{n}\} \cup \{0\}$ denote the semigroup of matrix units of class n whose multiplication is defined as follows:*

- : $0.(i, j) = (i, j).0 = 0$;
- : $(i, j)(k, l) = (i, l)$ when $j = k$;
- : $(i, j)(k, l) = 0$ when $j \neq k$.

Let x_0 and x_{ij}, y_{ij}, z_{ij} denote the free variables whose MU_n -degree are 0 and (i, j) , respectively. The following theorem addresses this issue:

Theorem 7.2. ([4], Theorem 4.9) *Let K be a field of characteristic zero. Then, the MU_n -graded identities of $M_n(K)$ follow from:*

$$\begin{aligned} & x_{ii}y_{ii} - y_{ii}x_{ii} \text{ when } i \in \widehat{n} \quad (5); \\ & x_{ij}y_{ji}z_{ij} - z_{ij}y_{ji}x_{ij} \text{ when } 1 \leq i, j \leq n \text{ } i \neq j \quad (6); \\ & x_0 \quad (7). \end{aligned}$$

Here, we extend this result for infinite integral domains. Let J_{MU_n} be the T_{MU_n} -ideal generated by (5), (6) and (7).

Let $h \in MU_n - \{0\}$. Let $W_h = \{w_h^{(1)}, \dots, w_h^{(n)}, \dots\}$ denote a countable set of commuting variables. Let $W = \bigcup_{h \in MU_n - \{0\}} W_h$ and let $\Omega_1 = K[W]$ be the polynomial ring in commuting variables of the set W .

Definition 7.3. A generic matrix of $M_n(\Omega_1)$ of MU_n -degree (i, j) is a homogeneous element of the following type:

$$A_{(i,j)}^{(k)} := w_{(i,j)}^{(k)} e_{ij} \text{ where } 1 \leq i, j \leq n \text{ and } k \in \{1, 2, \dots\}.$$

The MU_n -graded subalgebra generated by the generic matrices of $M_n(\Omega_1)$ is called the algebra of generic matrices which we denote by R_1 .

Notice that if m is a MU_n -graded monomial identity of R , then m follows from (7). The main steps of the proof of Lemmas 6.1 and 6.3 hold also for this grading and we obtain the following result.

Theorem 7.4. The MU_n -graded identities of R_1 follow from:

- $x_{ii}y_{jj} - y_{jj}x_{ii}$ when $i \in \widehat{n}$ (5);
- $x_{ij}y_{ji}z_{ij} - z_{ij}y_{ji}x_{ij}$ when $1 \leq i, j \leq n$, $i \neq j$ (6);
- x_0 (7).

If K is an infinite integral domain, then $T_{MU_n}(M_n(K)) = J_{MU_n}$.

In the rest of this paper we will only consider matrices over an infinite integral domain.

8. \mathbb{Z}_p -GRADED CENTRAL POLYNOMIAL OF $M_p(K)$

Now, we describe the \mathbb{Z}_p -graded central polynomials of $M_p(K)$ when p is a prime number.

Let $\pi : \mathbb{Z} \rightarrow \mathbb{Z}_n$ denote the canonical projection and let $j \in \mathbb{Z}_n$. The following convention will be described in this section:

$$y_{h,i}^j := y_{h,i}^k \text{ when } k = \pi^{-1}(j) \cap \widehat{n}.$$

Definition 8.1. ([8], Preliminaries) A sequence $(\gamma_1, \dots, \gamma_n)$ of elements of \mathbb{Z}_n is called a complete sequence when the following conditions are satisfied:

- $\gamma_1 + \dots + \gamma_n = 0$;
- $\{\gamma_1, \gamma_1 + \gamma_2, \dots, \gamma_1 + \dots + \gamma_n\} = \mathbb{Z}_n$.

The next lemma is the immediate consequence of the complete sequence definition.

Lemma 8.2. A sequence $(\gamma_1, \dots, \gamma_n)$ of elements of \mathbb{Z}_n is a complete sequence of \mathbb{Z}_n if and only if there exist matrix units $e_{i_1 j_1} \in M_n(K)_{\gamma_1}, \dots, e_{i_n j_n} \in M_n(K)_{\gamma_n}$ such that $i_{l+1} = j_l$ for all $(l+1) \in \widehat{n}$. Moreover, $\widehat{n} = \{i_1, \dots, i_n\}$ and $i_1 = j_n$.

Following word for word the proof of Brandão Júnior in [8] (see Lemma 1 and Proposition 1), we have the following lemma:

Lemma 8.3. The \mathbb{Z}_n -graded multilinear polynomial:

$$\sum_{\sigma \in H_n} x_{\sigma(1)} \cdots x_{\sigma(n)},$$

where $(\alpha(x_1), \dots, \alpha(x_n))$ is a complete sequence of \mathbb{Z}_n , is a \mathbb{Z}_n -graded central polynomial of $M_n(K)$. Furthermore, it is not a \mathbb{Z}_n -graded polynomial identity of $M_n(K)$.

Lemma 8.4. *Let $m = x_{i_1} \cdots x_{i_q}$ be a monomial such that $\alpha(m) = 0$ and $\alpha(x_{i_j}) = h_{i_j}$ for all $j \in \widehat{q}$. Let $y_{h_{i_1}, i_1}^{s_{i_1}^k} \cdots y_{h_{i_q}, i_q}^{s_{i_q}^k}$ be an entry of $A_{i_1} \cdots A_{i_q}$. If there exists a subsequence $(s_{i_{v_1}}^k, \dots, s_{i_{v_n}}^k)$ of $(s_{i_1}^k, \dots, s_{i_q}^k)$ such that $\{s_{i_{v_1}}^k, \dots, s_{i_{v_n}}^k\} = \widehat{n}$ and $s_{i_{v_1}}^k = s_{i_1}^k$, then there exist monomials m_1, \dots, m_n such that $(\alpha(m_1), \dots, \alpha(m_n))$ is a complete sequence of \mathbb{Z}_n and $m = m_1 m_2 \cdots m_n$.*

Proof. If $n = 1$, the proof is obvious. From now on, $n > 1$. First, we assume that m is a multilinear monomial.

In fact, there are matrix units $e_{j_1 l_1} \in M_n(K)_{\alpha(x_1)}, \dots, e_{j_q l_q} \in M_n(K)_{\alpha(x_q)}$ such that $e_{j_1 l_1} \cdots e_{j_q l_q} = e_{j_t l_t}$, where $j_t = s_{i_t}^k$ for all $t \in \widehat{q}$. Likewise, $j_1 = l_q$ because $m \in K\langle X \rangle_0$. We may suppose without any loss of generality that $j_t = s_{i_{v_t}}^k$ for $t = 2, \dots, n$.

Let: $m_i = x_i$, $i = 1, \dots, n-1$; $m_n = x_n \cdots x_q$.

Notice that $e_{j_1 l_1} e_{j_2 l_2} \cdots e_{j_n l_n} = e_{j_1 l_q}$. So $\{j_1, \dots, j_n\} = \widehat{n}$, $l_t = j_{t+1}$ for all $t \in \widehat{n-1}$ and $j_1 = l_q$. Consequently, by Lemma 8.2, it follows that m_1, \dots, m_n satisfy the thesis of this lemma.

Now, we assume that m is an arbitrary monomial. We would choose a multilinear monomial \overline{m} such that $h(m) = h(\overline{m})$. There exist monomials $\overline{m}_1, \dots, \overline{m}_n$ such that $(\alpha(\overline{m}_1), \dots, \alpha(\overline{m}_n))$ is a complete sequence of \mathbb{Z}_n and $\overline{m} = \overline{m}_1 \overline{m}_2 \cdots \overline{m}_n$. Thus, there must exist monomials m_1, \dots, m_n such that $m = m_1 \cdots m_n$ and $h(m_1) = h(\overline{m}_1), \dots, h(m_n) = h(\overline{m}_n)$. The proof is complete. \square

The proof of the following lemma is left as an exercise.

Lemma 8.5. *Let $A = \{a_1, \dots, a_l\} \subsetneq \mathbb{Z}_p$ be a set. Then:*

$$\{a_1 + i, \dots, a_l + i\} \neq \{a_1 + j, \dots, a_l + j\}$$

for any $i, j \in \mathbb{Z}_p$ distinct.

The next lemma is well known.

Lemma 8.6. *Let $z_1, z_2 \in K\langle X \rangle_1$. Then the monomials z_1^2 and $z_1^2 z_2^2$ are \mathbb{Z}_2 -graded central monomials of $M_2(K)$.*

Lemma 8.7. *Let $p > 2$. Let x_1, x_2 be variables such that $\alpha(x_1) = \alpha(x_2) \neq 0$. Then $(x_1 x_2)^p \equiv x_2^p x_1^p \pmod{T_{\mathbb{Z}_p}(M_p(K))}$.*

Proof. Let $A_1 \in R_{\alpha(x_1)}$ and $A_2 \in R_{\alpha(x_2)}$ be two generic matrices. By Lemma 3.2, it is obvious that all positions in the diagonal of $(A_1 A_2)^p$ (respectively $A_2^p A_1^p$) have non-zero entries. According to Lemma 6.1, it is sufficient to prove that $e_{11}(A_1 A_2)^p = e_{11}(A_2^p A_1^p)$.

In fact,

$$\begin{aligned} e_{11}(A_1 A_2)^p &= e_{11}(\sum_{i=1}^p y_{\alpha(x_1), 1}^i y_{\alpha(x_1), 2}^{i+\alpha(x_1)} e_{i\pi^{-1}(\overline{i+2\alpha(x_1)}) \cap \widehat{p}})^p = \\ &= (\prod_{i=1}^p (y_{\alpha(x_1), 1}^i y_{\alpha(x_1), 2}^{i+\alpha(x_1)})) e_{11} = (\prod_{i=1}^p y_{\alpha(x_1), 2}^{i+\alpha(x_1)}) (\prod_{i=1}^p y_{\alpha(x_1), 1}^i) e_{11} = \\ &= (\prod_{i=1}^p y_{\alpha(x_1), 2}^i) (\prod_{i=1}^p y_{\alpha(x_1), 1}^i) e_{11} = (A_2)^p e_{11} (A_1)^p e_{11} = A_2^p A_1^p e_{11}. \end{aligned}$$

So, $(x_1 x_2)^p \equiv x_2^p x_1^p \pmod{T_{\mathbb{Z}_p}(M_p(K))}$ as required. \square

Lemma 8.8. ([8], Lemma 8) *Let $p > 2$ and let $l \in \widehat{p-1}$. Let $m = x_1^p \cdots x_l^p$ be a \mathbb{Z}_p -graded monomial such that $\alpha(x_i) \neq \alpha(x_j)$ for $i \neq j$. Then $m \in C_{\mathbb{Z}_p}(M_p(K))$.*

Proof. First, we assume that $l = 1$.

According to Lemma 3.2

$$(A_1^{\alpha(x_1)})^p = \sum_{i=1}^p y_{\alpha(x_1),1}^i y_{\alpha(x_1),1}^{i+\alpha(x_1)} \cdots y_{\alpha(x_1),1}^{i+(p-1)\alpha(x_1)} e_{ii}.$$

So $y_{\alpha(x_1),1}^i y_{\alpha(x_1),1}^{i+\alpha(x_1)} \cdots y_{\alpha(x_1),1}^{i+(p-1)\alpha(x_1)} = y_{\alpha(x_1),1}^j y_{\alpha(x_1),1}^{j+\alpha(x_1)} \cdots y_{\alpha(x_1),1}^{j+(p-1)\alpha(x_1)}$ for all $i, j \in \widehat{p}$ because $\langle \alpha(x_1) \rangle = \mathbb{Z}_p$. Consequently $x_1^p \in C_{\mathbb{Z}_p}(M_p(K))$. Bearing in mind that $C_{\mathbb{Z}_p}(M_p(K))$ is a subalgebra, the result follows. \square

Definition 8.9. Let V_1 denote the $T_{\mathbb{Z}_p}$ -graded space generated by the monomials reported in the hypothesis of Lemma 8.6 or the monomials that satisfy the hypothesis of Lemma 8.8.

The proofs of the two following lemmas (Lemmas 8.10 and 8.11) are left as an exercise.

Lemma 8.10. Let $z_1, \dots, z_n \in K\langle X \rangle_1$. Let $m = z_1^{2.k_1} \cdots z_n^{2.k_n}$, where $k_1, \dots, k_n \in \mathbb{N} - \{0\}$. Then there exists $m_1 \in \langle V_1 \rangle^{T_{\mathbb{Z}_2}}$ such that $m - m_1 \equiv 0$ modulo $T_{\mathbb{Z}_2}(M_2(K))$.

Lemma 8.11. Let $p > 2$ and let $x_1, \dots, x_n \in K\langle X \rangle_i, i \neq 0$. Let $m = x_1^{p.k_1} \cdots x_n^{p.k_n}$ be a monomial, where $k_1, \dots, k_n \in \mathbb{N} - \{0\}$. Then there exists $m_1 \in \langle V_1 \rangle^{T_{\mathbb{Z}_p}}$ such that $m - m_1 \equiv 0$ modulo $T_{\mathbb{Z}_p}(M_p(K))$.

Lemma 8.12. Let $m = x_{i_1} \cdots x_{i_r} \in C_{\mathbb{Z}_p}(M_p(K))$. Then there exists $m_1 \in \langle V_1 \rangle^{T_{\mathbb{Z}_p}}$ such that $m - m_1 \equiv 0$ modulo $T_{\mathbb{Z}_p}(M_p(K))$.

Proof. Notice that $m \notin T_{\mathbb{Z}_p}(M_p(K))$. Moreover, at least one variable of m has \mathbb{Z}_p -degree different than 0.

By hypothesis, m is a \mathbb{Z}_p -graded central monomial of $M_p(K)$. Consequently, using the Lemma 3.2, it follows that:

$$y_{\alpha(x_{i_1}),i_1}^i y_{\alpha(x_{i_2}),i_2}^{i+\alpha(x_{i_1})} \cdots y_{\alpha(x_{i_r}),i_r}^{i+\alpha(x_{i_1}+\cdots+x_{i_{r-1}})} = y_{\alpha(x_{i_1}),i_1}^j y_{\alpha(x_{i_2}),i_2}^{j+\alpha(x_{i_1})} \cdots y_{\alpha(x_{i_r}),i_r}^{j+\alpha(x_{i_1}+\cdots+x_{i_{r-1}})}$$

for any $i, j \in \widehat{p}$.

Let x_{l_1}, \dots, x_{l_q} be all the different variables of the monomial m . k_i denotes the ordinary degree of m with respect to variable x_{l_i} . Note that, for each $i \in \widehat{q}$, k_i is a multiple of p .

Case 1: $\alpha(x_{l_i}) \neq 0$ for all $i \in \widehat{q}$. Let $x_{l_1}^{k_1} \cdots x_{l_q}^{k_q}$ be a monomial and let $A_{l_1} \in R_{\alpha(x_{l_1})}, \dots, A_{l_q} \in R_{\alpha(x_{l_q})}$ be generic matrices. Evidently, the matrices $A_{l_1}^{k_1} \cdots A_{l_q}^{k_q}$ and $m(A_{l_1}, \dots, A_{l_q})$ have in position $(1, 1)$, the same non-zero entry. Therefore, by Lemma 6.1, $m \equiv x_{l_1}^{k_1} \cdots x_{l_q}^{k_q} \pmod{J_1}$. Applying Lemma 8.10 or Lemma 8.11, we are done.

Case 2: there exists $i \in \widehat{q}$ such that $\alpha(x_{l_i}) = 0$. Suppose that all variables of G -degree 0 are $\{x_{l_1}, \dots, x_{l_s}\}$. Choose $m_1 = (x_{l_{s+1}}^{k_{l_{s+1}}} \cdots x_{l_s}^{k_s})^p x_{l_{s+1}}^{k_{s+1}-p} x_{l_{s+2}}^{k_{s+2}} \cdots x_{l_q}^{k_{l_q}}$. Evidently, $m_1(A_{l_1}, \dots, A_{l_q})$ and $m(A_{l_1}, \dots, A_{l_r})$ have in position $(1, 1)$ the same non-zero entry. Applying the ideas of previous case, the result follows. \square

Lemma 8.13. Let $m = x_{i_1} \cdots x_{i_q} \in K\langle X \rangle_0 - (C_{\mathbb{Z}_p}(M_p(K))) \cap (K\langle X \rangle_0)$. All entries in the diagonal of $A_{i_1} \cdots A_{i_q}$ are non-zero and pairwise distinct.

Proof. (Sketches) According to Lemma 3.2, all entries in the diagonal are non-zero. If $p = 2$, the analysis is obvious.

Henceforth, suppose that $p > 2$. By hypothesis, $x_{i_1} \cdots x_{i_q} \in K\langle X \rangle_0 - C_{\mathbb{Z}_p}(M_p(K)) \cap K\langle X \rangle_0$. Thus, the following condition is satisfied: there exist $j_1 < \cdots < j_l \in \widehat{q}$, where $x_{i_{j_1}} = \cdots = x_{i_{j_l}}$ and $x_{i_t} \neq x_{i_{j_1}}$ for all $t \in \widehat{q} - \{j_1, \dots, j_l\}$ such that:

$$\begin{aligned} & y_{\alpha(x_{i_{j_1}}), i_{j_1}}^{k_1 + \alpha(x_{i_1}) + \cdots + \alpha(x_{i_{j_1-1}})} \cdots y_{\alpha(x_{i_{j_l}}), i_{j_l}}^{k_1 + \alpha(x_{i_1}) + \cdots + \alpha(x_{i_{j_l-1}})} \neq \\ & y_{\alpha(x_{i_{j_1}}), i_{j_1}}^{k_2 + \alpha(x_{i_1}) + \cdots + \alpha(x_{i_{j_1-1}})} \cdots y_{\alpha(x_{i_{j_l}}), i_{j_l}}^{k_2 + \alpha(x_{i_1}) + \cdots + \alpha(x_{i_{j_l-1}})}. \end{aligned}$$

By Lemma 8.5 and some calculations, we may conclude that:

$$\begin{aligned} & y_{\alpha(x_{i_1}), i_1}^{q_1 + \alpha(x_{i_1}) + \cdots + \alpha(x_{i_{j_1-1}})} \cdots y_{\alpha(x_{i_{j_l}}), i_{j_l}}^{q_1 + \alpha(x_{i_1}) + \cdots + \alpha(x_{i_{j_l-1}})} \neq \\ & y_{\alpha(x_{i_1}), i_1}^{q_2 + \alpha(x_{i_1}) + \cdots + \alpha(x_{i_{j_1-1}})} \cdots y_{\alpha(x_{i_{j_l}}), i_{j_l}}^{q_2 + \alpha(x_{i_1}) + \cdots + \alpha(x_{i_{j_l-1}})} \text{ for all } q_1 \neq q_2 \in \mathbb{Z}_p. \end{aligned}$$

The proof is complete. \square

In what follows, we use substantially the proof of Theorem 6 used by Brandão Júnior [8].

Theorem 8.14. *The \mathbb{Z}_p -graded central polynomials of $M_p(K)$ follow from:*

$$z_1(x_1x_2 - x_2x_1)z_2 \text{ when } \alpha(x_1) = \alpha(x_2) = \bar{0} \quad (8);$$

$$z_1(x_1x_2x_3 - x_3x_2x_1)z_2 \text{ when } \alpha(x_1) = -\alpha(x_2) = \alpha(x_3) \neq \bar{0} \quad (9);$$

$$\text{The monomials cited in the definition 8.9} \quad (10);$$

$$\sum_{\sigma \in H_p} x_{\sigma(1)} \cdots x_{\sigma(p)}, \text{ where } (\alpha(x_1), \dots, \alpha(x_p)) \text{ is a complete sequence of } \mathbb{Z}_p \quad (11).$$

The monomials $z_1, z_2 \in \bigcup_{g \in \mathbb{Z}_p} X_g$.

Proof. Let W be the $T_{\mathbb{Z}_p}$ -space generated by (8), (9), (10) and (11). We prove that $C_{\mathbb{Z}_p}(M_p(K)) \subset W$. Let $f(x_1, \dots, x_q) = \sum_{i=1}^l \lambda_i m_i \in C_{\mathbb{Z}_p}(M_p(K)) - T_{\mathbb{Z}_p}(M_p(K))$. By Lemma 3.8 and Corollary 3.11, we may assume that f is a multi-homogeneous polynomial. We may suppose that $\alpha(m_1) = \cdots = \alpha(m_l) = 0$, $m_i - m_j$ is not an element of $T_{\mathbb{Z}_p}(M_p(K))$ and each $m_i \notin C_{\mathbb{Z}_p}(M_p(K))$ (Lemma 8.12).

Let $A_1 \in R_{\alpha(x_1)}, \dots, A_q \in R_{\alpha(x_q)}$ be generic matrices. So $f(A_1, \dots, A_q) = \text{diag}(F_1, \dots, F_p)$, where $F_1 = \cdots = F_p \neq 0$. By Lemma 8.13, for each $j \in \widehat{p}$, all positions in the diagonal of the matrix $m_j(A_1, \dots, A_q)$ have non-zero entries. Furthermore, these entries are pairwise distinct.

Reordering the indices, if necessary, there exist $1 \leq i_1 < \cdots < i_p \leq l$ such that $\lambda_{i_1} = \cdots = \lambda_{i_p} \neq 0$ and $e_{11}m_{i_1}(A_1, \dots, A_q)e_{11} = e_{1l_2}m_{i_{l_2}}(A_1, \dots, A_q)e_{l_21}$ for all $l_2 \in \widehat{p} - \{1\}$. Assume that $m_{i_1} = x_{j_1} \cdots x_{j_s}$ and the entry in position (1,1) of $m_{i_1}(A_1, \dots, A_q)$ is $y_{\alpha(x_{j_1}), j_1}^{a_1} \cdots y_{\alpha(x_{j_s}), j_s}^{a_s}$. Notice that the multi-set $\{a_1, \dots, a_s\}$ contains \widehat{p} .

According to Lemma 8.4, there are monomials r_1, \dots, r_p such that $m_{i_1} = r_1 \cdots r_p$ where $(\alpha(r_1), \dots, \alpha(r_p))$ is a complete sequence of \mathbb{Z}_p . For each $j \in \widehat{p}$, there is a unique permutation $\sigma \in H_p$ such that the matrices:

$$m_{i_j}(A_1, \dots, A_q) \text{ and } r_{\sigma(1)} \cdots r_{\sigma(p)}(A_1, \dots, A_q)$$

have, in the position (1,1), the same non-zero entry. By Lemma 6.1:

$$m_{i_j} \equiv r_{\sigma(1)} \cdots r_{\sigma(p)} \text{ mod } T_{\mathbb{Z}_p}(M_p(K)).$$

According to Lemma 8.3, it is clear that:

$$g(x_1, \dots, x_r) = \lambda_{i_1}(\sum_{\sigma \in H_p} r_{\sigma(1)} \cdots r_{\sigma(p)}) \in W.$$

Then, $f - g \equiv f - \lambda_{i_1}(m_{i_1} + m_{i_2} + \cdots + m_{i_p})$ modulo $T_{\mathbb{Z}_p}(M_p(K))$. If $l - p = 0$, it follows that $f \in W$. If $l - p \geq p$ or $1 \leq l - p \leq p - 1$, the same argument can be repeated. From an inductive argument on l , the result follows. \square

9. \mathbb{Z} -GRADED CENTRAL POLYNOMIALS OF $M_n(K)$

In this section, we use the same technique as in previous section to detail a script of the proof. The first Lemma is similar to Lemma 8.13.

Lemma 9.1. *Let $x_{i_1}, \dots, x_{i_m} \in K\langle X \rangle_0$.*

If $A_{i_1}^{\alpha(x_{i_1})} \cdots A_{i_m}^{\alpha(x_{i_m})}$ is a non-zero matrix, then all non-zero entries of that matrix are pairwise distinct.

Proof. (Sketches) If only one position in $A_{i_1} \cdots A_{i_m}$ has a non-zero entry, the proof is obvious. Suppose that there exist, at least, two positions $(k_1, k_1), (k_2, k_2) \in \hat{n} \times \hat{n}$ such that:

$$e_{1k_1}(A_{i_1} \cdots A_{i_m})e_{k_11}, e_{1k_2}(A_{i_1} \cdots A_{i_m})e_{k_21} \neq 0.$$

Our aim is to prove that:

$$e_{1k_1}(A_{i_1} \cdots A_{i_m})e_{k_11} \neq e_{1k_2}(A_{i_1} \cdots A_{i_m})e_{k_21}.$$

Suppose by contradiction that:

$$e_{1k_1}(A_{i_1} \cdots A_{i_m})e_{k_11} = e_{1k_2}(A_{i_1} \cdots A_{i_m})e_{k_21}.$$

By contradiction, the result follows. \square

The main steps of the proof of the Lemmas 8.3, Lemma 8.4 and Theorem 8.14 hold also for this grading and we obtain the following result.

Theorem 9.2. *Let K be an infinite integral domain. Then the \mathbb{Z} -graded central polynomials of $M_n(K)$ follow from:*

$$z_1(x_1x_2 - x_2x_1)z_2 \text{ when } \alpha(x_1) = \alpha(x_2) = 0 \quad (12);$$

$$z_1(x_1x_2x_3 - x_3x_2x_1)z_2 \text{ when } \alpha(x_1) = -\alpha(x_2) = \alpha(x_3) \neq 0 \quad (13);$$

$$z_1x_1z_2 \text{ when } |\alpha(x_1)| \geq n \quad (14);$$

$$\sum_{\sigma \in H_n} x_{\sigma(1)} \cdots x_{\sigma(n)}, \text{ where } (\overline{\alpha(x_1)}, \dots, \overline{\alpha(x_n)}) \text{ is a complete sequence of } \mathbb{Z}_n \text{ and } |\alpha(x_i)| < n \quad (15).$$

The monomials $z_1, z_2 \in \bigcup_{i \in \mathbb{Z}} X_i$.

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